Queueing Systems 33 (1999) 177–204

Heavy-traffic analysis for the GI/G/1 queue with heavy-tailed distributions

O.J. Boxma [a] and J.W. Cohen [b]

[a] *Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands*
E-mail: boxma@win.tue.nl
[b] *CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands*

Received September 1997; revised January 1999

We consider a GI/G/1 queue in which the service time distribution and/or the interarrival time distribution has a heavy tail, i.e., a tail behaviour like $t^{-\nu}$ with $1 < \nu \leqslant 2$, so that the mean is finite but the variance is infinite. We prove a heavy-traffic limit theorem for the distribution of the stationary actual waiting time **W**. If the tail of the service time distribution is heavier than that of the interarrival time distribution, and the traffic load $a \to 1$, then **W**, multiplied by an appropriate 'coefficient of contraction' that is a function of a, converges in distribution to the Kovalenko distribution. If the tail of the interarrival time distribution is heavier than that of the service time distribution, and the traffic load $a \to 1$, then **W**, multiplied by another appropriate 'coefficient of contraction' that is a function of a, converges in distribution to the negative exponential distribution.

Keywords: GI/G/1 queue, heavy tails, regular variation, waiting time distribution, heavy-traffic limit theorems

1. Introduction

In this paper we consider the classical GI/G/1 queue, with i.i.d. (independent, identically distributed) interarrival times $\sigma_1, \sigma_2, \ldots$ with distribution $A(\cdot)$ with finite mean α, and with i.i.d. service times $\tau_1, \tau_2, \ldots$ with distribution $B(\cdot)$ with finite mean β. The traffic load $a := \beta/\alpha$ is assumed to be less than one, so that the queue is stable.

When the variances of the interarrival and service time distributions are finite, the standard heavy-traffic limit theorem for the stationary actual waiting time **W** in the GI/G/1 queue holds, i.e.,

$$\lim_{\Delta \downarrow 0} \mathbf{P}[\Delta \mathbf{W} \leqslant t] = 1 - \mathrm{e}^{-t}, \quad t \geqslant 0, \tag{1.1}$$

with $\Delta := 2(\alpha - \beta)/(\mathrm{Var}(\sigma_1) + \mathrm{Var}(\tau_1))$. This exponential heavy-traffic theorem was obtained by Kingman in the early sixties; see Kingman [27] for an early survey, and Whitt [34] for an extensive overview of heavy-traffic limit theorems for queues.

© J.C. Baltzer AG, Science Publishers

In the present study we prove a heavy-traffic limit theorem for the GI/G/1 queue in which the second moment of the service time and/or interarrival time is *infinite*. Our main motivation for this study, apart from the wish to extend the classical GI/G/1 theory, is the close relation between the ordinary GI/G/1 queue and the so-called fluid queue, coupled with the recent interest in fluid queues with input distributions that have an *infinite variance*. Let us elaborate. Plots of recent traffic measurements in Ethernet Local Area Networks [35], Wide Area Networks [32] and VBR video [2] have shown a striking similarity when one considers a time period of hours, minutes or milliseconds: bursty subperiods are alternated by less bursty subperiods on each time scale. This scale-invariant or *self-similar* feature of traffic, and the related phenomenon of *long-range dependence* (i.e., the integral of the covariance of the traffic rate diverges), were convincingly demonstrated in [28]. A natural possibility to introduce long-range dependence (LRD) in a traffic process is to take a fluid queue fed by one or more on/off sources (viz., sources that alternate between active and silent periods), and to assume that either the on-period or the off-period of a source has the following 'heavy-tail' behaviour:

$$\mathbf{P}[\mathbf{O} > t] \overset{t\to\infty}{\sim} h_\nu t^{-\nu}, \tag{1.2}$$

with h_ν a positive constant and $1 < \nu < 2$, giving rise to an infinite variance (here $f(t) \overset{t\to\infty}{\sim} g(t)$ stands for $f(t)/g(t) \to 1$ with $t \to \infty$). As soon as one of the sources exhibits such behaviour, the cumulative input process is LRD. As observed in [35], in many cases on- and/or off-periods of actual traffic sources do indeed exhibit such a heavy-tail behaviour. The occurrence of heavy-tailed on- and/or off-periods of sources seems to provide the most natural explanation of LRD and self-similarity in aggregated packet traffic. These observations have triggered much research on fluid queues with, in particular, heavy-tailed on-period distributions. In this context, regularly varying and subexponential on-period distributions have received special attention.

There appears to be a strong relation, and a considerable similarity of behaviour, between the fluid queue and the ordinary single server queue. See, e.g., [9,25]. The latter paper studies a fluid queue fed by independent on/off sources, and relates the buffer content, embedded at the beginnings of periods in which at least one source is active, to the waiting time in a certain G/G/1 queue. Among other things, this allows one to exploit [4,5] a result of [8] which states that, in the ordinary GI/G/1 queue, the tail of the waiting time distribution is regularly varying if and only if the tail of the service time distribution is regularly varying.

The above elaboration on the intricate phenomena encountered in fluid models clearly motivates a study of the GI/G/1 model with heavy tails. While the results to be obtained have their own merit, they also should provide a better insight into fluid models.

Generally speaking, it is very difficult to obtain explicit waiting-time results for the GI/G/1 queue with heavy-tailed interarrival and/or service time distribution. However, we are able to obtain *heavy-traffic* results for the actual waiting time, and these results give much insight into the behaviour of the single-server queue with heavy-tailed distributions. We consider service time and/or interarrival time distributions for which the dominant tail behaviour is a generalization of the behaviour specified in (1.2). First assume that the tail of the service time distribution exhibits this behaviour, with $1 < \nu < 2$, while the tail of the interarrival time distribution is 'less heavy' than that of the service time distribution. Our main result (theorem 5.1) for this case states the following. The 'contracted' waiting time $\Delta(a)\mathbf{W}/\beta$ converges in distribution for $a \uparrow 1$ to a limiting distribution $R_{\nu-1}(t)$. This distribution is specified in (5.14), and the 'coefficient of contraction' $\Delta(a)$ (that typically behaves roughly like $(1-a)^{1/(\nu-1)}$) is specified in the 'contraction equation' (4.6).

Next assume that the tail of the interarrival time distribution exhibits the tail behaviour of (1.2), while the tail of the service time distribution is less heavy than that of the interarrival time distribution. Our main result (theorem 7.1) for this case states the following. The 'contracted' waiting time $\Lambda(a)\mathbf{W}/\alpha$ converges in distribution for $a \uparrow 1$ to the negative exponential distribution. The 'coefficient of contraction' $\Lambda(a)$ (that typically behaves roughly like $(1-a)^{1/(\nu-1)}$) is specified in (7.5).

Remark 1.1. Recently some special cases of theorem 5.1 have been obtained. In [12] this heavy-traffic result has been derived for the M/G/1 queue with a special Pareto-type service time distribution; in [13] it has been obtained for the GI/G/1 queue with a Pareto-type service time distribution and an interarrival time distribution that has a less heavy tail than the service time distribution. The report [14] extends the work of the present paper to handle the case in which the tails of the interarrival time and service time distributions are 'similarly heavy'.

Remark 1.2. The above-mentioned heavy-traffic limit theorems open possibilities for approximating the waiting time distribution in the GI/G/1 queue with heavy-tailed interarrival and/or service time distributions. These possibilities are explored in [6] and appear to be very promising. For example, approximating $\mathbf{P}[\mathbf{W} > t]$ in the case of theorem 5.1 by $1 - R_{\nu-1}(\Delta(a)t/\beta)$ gives, for $\nu = 3/2$, a remarkably accurate approximation, even if the traffic load a is not close to one; cf. remark 5.5.

The remainder of the paper is organized in the following way. Section 2 presents the class of service time distributions under consideration in the case of theorem 5.1; examples taken from this class are given in section 3. In section 4 the coefficient of contraction $\Delta(a)$ is discussed. Theorem 5.1 is proven in section 5. Section 6 presents the class of interarrival time distributions under consideration in the case of theorem 7.1. In section 7 the coefficient of contraction $\Lambda(a)$ is discussed, after which theorem 7.1 is proven.

Our proofs rely on a well-known expression for the Laplace–Stieltjes transform of the GI/G/1 waiting time distribution, on representations of the Laplace–Stieltjes transform of heavy-tailed distributions, and on boundary value techniques. Various analytical results and derivations concerning these are gathered in the appendices.

2. On the service and interarrival distributions

For the GI/G/1 queueing model, denote by $A(t)$ the distribution of the interarrival times σ_n, and by $B(t)$ that of the service times τ_n. In this section we describe the classes of distributions $A(\cdot)$ and $B(\cdot)$ for which we analyse the heavy-traffic behaviour of the waiting time distribution.

Put

$$\alpha := \int_0^\infty t\,\mathrm{d}A(t) < \infty, \qquad \beta := \int_0^\infty t\,\mathrm{d}B(t) < \infty, \qquad a := \beta/\alpha < 1, \tag{2.1}$$

and, for $\mathrm{Re}\,\rho \geqslant 0$:

$$\alpha\{\rho\} := \int_{0-}^\infty \mathrm{e}^{-\rho t}\,\mathrm{d}A(t), \qquad \beta\{\rho\} := \int_{0-}^\infty \mathrm{e}^{-\rho t}\,\mathrm{d}B(t). \tag{2.2}$$

Concerning the service time distribution $B(\cdot)$ we only introduce assumptions about its tail probabilities, i.e., about $1 - B(t)$ for $t \to \infty$. It is assumed that for some finite T we may write: for $t > T$,

$$1 - B(t) = G_1(t) + G_2(t). \tag{2.3}$$

Here $G_2(t)$ will be a function such that: for a $\delta > 0$,

$$\int_T^\infty \mathrm{e}^{-\rho t} G_2(t)\,\mathrm{d}t \quad \text{exists for Re } \rho > -\delta. \tag{2.4}$$

For instance, (2.4) holds if

$$G_2(t) = \mathrm{O}\big(\mathrm{e}^{-\delta t}\big) \quad \text{for } t \to \infty. \tag{2.5}$$

Note that it is no restriction to take $T = \beta$, as we shall do in the sequel for reasons of simplicity.

The function $G_1(t)$ shall describe the dominant asymptotic behaviour of $1 - B(t)$; e.g., $G_1(t) = C(\beta/t)^{3/2}$, with C a positive constant. In section 3 we shall discuss the class of functions $G_1(t)$ in more detail. We shall not work with the representation (2.3) but mainly with the following closely related LST (Laplace–Stieltjes) representation of $\beta\{\rho\}$. We have chosen this representation because it easily relates to the tail behaviour of probability distributions, and because it allows us to operate in the whole complex half-plane; see further the examples in section 3. It is assumed that $\beta\{\rho\}$ can be represented as: for $\mathrm{Re}\,\rho \geqslant 0$,

$$1 - \frac{1 - \beta\{\rho\}}{\beta\rho} = g(\beta\rho) + c(\beta\rho)^{\nu - 1} L(\beta\rho), \tag{2.6}$$

where:

$$c > 0 \text{ is a constant;} \tag{2.7a}$$

$$1 < \nu \leqslant 2; \tag{2.7b}$$

$$g(\beta\rho) \text{ is a regular function of } \rho \text{ for } \mathrm{Re}\,\rho > -\delta,\ g(0) = 0; \tag{2.7c}$$

$$\begin{cases} L(\beta\rho) \text{ is regular for } \operatorname{Re}\rho > 0, \text{ and continuous for } \operatorname{Re}\rho \geqslant 0, \\ \quad \text{except possibly at } \rho = 0, \\ L(\beta\rho) \to b > 0 \quad \text{for } |\rho| \to 0,\ \operatorname{Re}\rho \geqslant 0, \text{ with } b = \infty \text{ if } \nu = 2, \\ \lim\limits_{x \downarrow 0} \dfrac{L(\beta\rho x)}{L(x)} = 1 \quad \text{for } \operatorname{Re}\rho \geqslant 0,\ \rho \neq 0; \end{cases} \tag{2.7d}$$

$$\text{for } \lambda \in (1, \nu): \quad \int_0^\infty t^\lambda \, \mathrm{d}B(t) < \infty. \tag{2.7e}$$

Concerning $A(t)$ it will be assumed that

$$M_\mu := \int_0^\infty t^\mu \, \mathrm{d}A(t) < \infty \quad \text{for a } \mu > \nu. \tag{2.8}$$

Remark 2.1. Whenever in the present analysis many-valued functions occur, like $(\beta\rho)^\nu$ or $\log \beta\rho$, then they are assumed to be defined by their principal value; the principal value of $(\beta\rho)^\nu$ is positive for $\beta\rho > 0$, that of $\log \beta\rho$ is real for $\beta\rho > 0$.

Remark 2.2. The class of service time distributions specified by (2.6) and (2.7) contains the class of regularly varying distributions of index $-\nu \in (-2, -1)$; cf. [3, pp. 333, 334], and notice that $L(x)$ is a slowly varying function of x for x real. In particular, it follows from [3, theorem 8.1.6, pp. 333, 334] that $1 - B(t)$ is regularly varying of index $-\nu \in (-2, -1)$ iff $1 - (1 - \beta\{\rho\})/(\beta\rho) \sim$ const. $\rho^{\nu-1} L(\beta\rho)$, with $L(\cdot)$ a slowly varying function. It should also be noticed that condition (2.7e) for $\nu < 2$ is immediately implied by the previous conditions; for reasons of reference we have inserted it in (2.7).

3. On the class of distributions $B(\cdot)$

In this section we present a number of examples of service time distributions with heavy tails, of which the LSTs have the properties described in the previous section.

Case (i)

In [12] the subject of study is the M/G/1 queue with service time distribution $B(t)$ given by

$$\begin{aligned} &1 - B(t) = \frac{s^{2-\nu}}{\Gamma(2-\nu)} \widehat{\delta} \int_0^\infty \mathrm{e}^{-s\theta} \frac{\theta}{(\theta + t)^\nu} \, \mathrm{d}\theta, \quad t \geqslant 0, \\ &1 < \nu < 2, \quad s > 0, \quad 0 < \widehat{\delta} \leqslant 1, \\ &\beta = \frac{2-\nu}{\nu - 1} \frac{\widehat{\delta}}{s}, \qquad B(0+) = 1 - \widehat{\delta}. \end{aligned} \tag{3.1}$$

It is readily seen that (3.1) implies

$$1 - B(t) \sim c\left(\frac{\beta}{t}\right)^{\nu} \quad \text{for } t \to \infty, \quad c > 0,$$

and it is also readily verified that this $B(t)$ for $t > \beta$ can be written in the form (2.3). In [12] it has been shown that (3.1) leads to: for $\operatorname{Re} \rho \geqslant 0$,

$$1 - \frac{1 - \beta\{\rho\}}{\beta\rho} = -\frac{\rho/s}{1 - \rho/s} - \frac{1}{2 - \nu}\frac{\rho/s}{(1 - \rho/s)^2} + \frac{1}{2 - \nu}\frac{(\rho/s)^{\nu-1}}{(1 - \rho/s)^2}. \tag{3.2}$$

Obviously, (3.2) satisfies (2.6) and (2.7); note that here

$$L(\beta\rho) = \frac{1 - (\rho/s)^{2-\nu}[1 + (2 - \nu)(1 - \rho/s)]}{(1 - \rho/s)^2}.$$

Further, note that for $\nu = 3/2$, $\widehat{\delta} = 1$,

$$\frac{1 - \beta\{\rho\}}{\beta\rho} = \frac{1}{(1 + \sqrt{\beta\rho})^2}, \quad \operatorname{Re} \rho \geqslant 0. \tag{3.3}$$

Case (ii)

In [13] the following case is considered:

$$1 - B(t) = c\left(\frac{\beta}{t}\right)^{\nu} + G_2(t) \quad \text{for } t > \beta, \quad c > 0,\ 1 < \nu < 2, \tag{3.4}$$

and it is shown that: for $\operatorname{Re} \rho \geqslant 0$,

$$1 - \frac{1 - \beta\{\rho\}}{\beta\rho} = g_1(\beta\rho) + \frac{c\pi}{\Gamma(\nu)\sin(\nu - 1)\pi}(\beta\rho)^{\nu-1}, \tag{3.5}$$

with: for $\operatorname{Re} \rho > -\delta$,

$$\begin{aligned} g_1(\beta\rho) = &\int_0^{\beta} \left(1 - \mathrm{e}^{-\rho t}\right)\left(1 - B(t)\right)\frac{\mathrm{d}t}{\beta} + \int_{\beta}^{\infty} c\left(\frac{\beta}{t}\right)^{\nu}\frac{\mathrm{d}t}{\beta} \\ &+ \int_{\beta}^{\infty} \left(1 - \mathrm{e}^{-\rho t}\right)G_2(t)\frac{t}{\beta} + \frac{c}{\nu - 1}\mathrm{e}^{-\beta\rho} \\ &+ \frac{c\rho}{1 - \nu}\int_0^{\beta} \mathrm{e}^{-\rho t}\left(\frac{t}{\beta}\right)^{1-\nu} \mathrm{d}t; \end{aligned} \tag{3.6}$$

obviously, $g_1(0) = 0$ and $g_1(\beta\rho)$ is regular for $\operatorname{Re} \rho > -\delta$. Clearly, (3.5) satisfies (2.6) and (2.7); note that here $L(\beta\rho)$ is a constant.

The following extension is also considered in [13]:

$$\begin{aligned} &1 - B(t) = c\left(\frac{\beta}{t}\right)^{\nu} + \sum_{n=1}^{N} c_n\left(\frac{\beta}{t}\right)^{\nu_n} + G_2(t) \quad \text{for } t > \beta, \\ &c > 0, \quad 1 < \nu < 2, \quad \nu_n > \nu, \quad 1 \leqslant N < \infty. \end{aligned} \tag{3.7}$$

From [13] it is seen that for this $B(t)$ we again get (2.6) and (2.7), but of course with a different $g(\cdot)$ and a different $L(\beta\rho)$. Actually, we have for $\operatorname{Re}\rho \geqslant 0$:

$$1 - \frac{1-\beta\{\rho\}}{\beta\rho} = g_2(\beta\rho) + \frac{c\pi(\beta\rho)^{\nu-1}}{\Gamma(\nu)\sin(\nu-1)\pi}\left\{1 + \frac{c_1}{c}\frac{\Gamma(\nu)\sin(\nu-1)\pi}{\Gamma(\nu_1)\sin(\nu_1-1)\pi}(\beta\rho)^{\nu_1-\nu}\right\}, \quad (3.8)$$

for $c_n = 0$, $n = 2, \ldots, N$, $\nu_1 > \nu$, ν_1 not an integer; whereas for $\nu_1 = k > \nu$, k an integer,

$$1 - \frac{1-\beta\{\rho\}}{\beta\rho} = g_3(\beta\rho) + \frac{c\pi(\beta\rho)^{\nu-1}}{\Gamma(\nu)\sin(\nu-1)\pi} \times \left\{1 + \frac{c_1}{c}(-1)^{k-1}\frac{\Gamma(\nu)\sin(\nu-1)\pi}{\Gamma(k-1)}(\beta\rho)^{k-\nu}\log\beta\rho\right\}. \quad (3.9)$$

Case (iii)

The case

$$1 - B(t) = c\left(\frac{\beta}{t}\right)^{\nu}\left\{\log\frac{t}{\beta} - \frac{\Gamma^{(1)}(1-\nu)}{\Gamma(1-\nu)}\right\} + G_2(t) \quad \text{for } t \geqslant \beta, \quad (3.10)$$

with $c > 0$, $1 < \nu < 2$. From [18, Vol. I, p. 469], it is seen that: for $\operatorname{Re}\rho \geqslant 0$,

$$1 - \frac{1-\beta\{\rho\}}{\beta\rho} = g_4(\beta\rho) + c\Gamma(1-\nu)(\beta\rho)^{\nu-1}\log\beta\rho, \quad (3.11)$$

and this agrees again with (2.6) and (2.7); note that

$$c\Gamma(1-\nu) = -\frac{c\pi}{\Gamma(\nu)\sin(\nu-1)\pi},$$

so that for the present case

$$L(\beta\rho) = \frac{\pi}{\Gamma(\nu)\sin(\nu-1)\pi}\log\frac{1}{\beta\rho}, \quad \operatorname{Re}\rho \geqslant 0,\ \rho \neq 0.$$

Case (iv)

The case

$$1 - B(t) = c\left(\frac{\beta}{t}\right)^2 + G_2(t), \quad t > \beta. \quad (3.12)$$

From [18, Vol. I, p. 467], it is seen that: for $\operatorname{Re}\rho \geqslant 0$,

$$1 - \frac{1-\beta\{\rho\}}{\beta\rho} = g_5(\beta\rho) + c\beta\rho\log\frac{1}{\beta\rho}, \quad (3.13)$$

with $g_5(\beta\rho)$ regular for $\operatorname{Re}\rho > -\delta$ and $g_5(0) = 0$. Obviously, we have here an example with $\nu = 2$ and

$$L(\beta\rho) = \log \frac{1}{\beta\rho}. \tag{3.14}$$

A heavy-tailed distribution of the type (3.12) has been studied in [1].

Case (v)

The case

$$1 - B(t) = c\left(\frac{\beta}{t}\right)^2 \log \frac{t}{\beta} + G_2(t), \quad t \geqslant \beta. \tag{3.15}$$

From (3.15) we have: for $\operatorname{Re}\rho \geqslant 0$,

$$1 - \frac{1 - \beta\{\rho\}}{\beta\rho} = h_1(\beta\rho) - c\int_\beta^\infty \mathrm{e}^{-\rho t}\left(\frac{\beta}{t}\right)^2 \log \frac{t}{\beta}\frac{\mathrm{d}t}{\beta}, \tag{3.16}$$

with

$$\begin{aligned} h_1(\beta\rho) := &\int_0^\beta \left(1 - \mathrm{e}^{-\rho t}\right)\left(1 - B(t)\right)\frac{\mathrm{d}t}{\beta} \\ &+ \int_\beta^\infty \left(1 - \mathrm{e}^{-\rho t}\right) G_2(t)\frac{\mathrm{d}t}{\beta} + c\int_\beta^\infty \left(\frac{\beta}{t}\right)^2\left(\log \frac{t}{\beta}\right)\frac{\mathrm{d}t}{\beta}. \end{aligned} \tag{3.17}$$

Obviously, $h_1(\beta\rho)$ is regular for $\operatorname{Re}\rho > -\delta$, since the Laplace transform of $G_2(t)$ exists for $\operatorname{Re}\rho > -\delta$, cf. (2.4).

The integral in (3.16) is calculated in appendix A. We obtain from (3.16), (A.1), (A.6) and (A.7): for $\operatorname{Re}\rho \geqslant 0$,

$$1 - \frac{1 - \beta\{\rho\}}{\beta\rho} = g_6(\beta\rho) + c(\beta\rho)\log \frac{1}{\beta\rho} + \frac{1}{2}c(\beta\rho)^2(\log \gamma\beta\rho)^2, \tag{3.18}$$

with

$$g_6(\beta\rho) := h_1(\beta\rho) - ch_2(\beta\rho); \tag{3.19}$$

for $h_2(\beta\rho)$ see (A.7), and it is readily seen that

$$\begin{aligned} &g_6(\beta\rho) \text{ is regular for } \operatorname{Re}\rho > -\delta \text{ and } g_6(0) = 0; \\ &L(\beta\rho) = \left(\log \frac{1}{\beta\rho}\right)\left[1 - \frac{1}{2}\beta\rho\frac{(\log \gamma\beta\rho)^2}{\log \beta\rho}\right], \quad \operatorname{Re}\rho \geqslant 0,\ \rho \neq 0. \end{aligned} \tag{3.20}$$

4. The coefficient of contraction

In the heavy-traffic limit theorem, to be presented in section 5, it will be shown that the waiting time **W**, scaled by the *coefficient of contraction* $\Delta(a)$, approaches a proper limiting distribution for $a \uparrow 1$. In the present section that coefficient of contraction is discussed.

The function

$$1 - \frac{1 - \beta\{\rho\}}{\beta\rho}, \quad \rho \geqslant 0,$$

is zero for $\rho = 0$. It is monotonically increasing in ρ with limit equal to one for $\rho \to \infty$. Hence the equation

$$1 - \frac{1 - \beta\{\rho\}}{\beta\rho} = \frac{1-a}{a}, \quad \rho > 0, \ a \in \left(\frac{1}{2}, 1\right), \tag{4.1}$$

has a unique root $\rho = \delta(a)/\beta > 0$ and

$$\delta(a) \downarrow 0 \quad \text{for } a \uparrow 1. \tag{4.2a}$$

$$\delta(a) \text{ is regular for } a \in \left(\tfrac{1}{2}, 1\right), \text{ continuous for } a \in \left(\tfrac{1}{2}, 1\right]. \tag{4.2b}$$

Consequently, cf. (2.6), $\delta(a)$ is that root of

$$g(x) + cx^{\nu-1}L(x) = \frac{1-a}{a}, \quad x > 0, \ a \in \left(\frac{1}{2}, 1\right), \tag{4.3}$$

which satisfies (4.2a). From (2.7c) it is seen that

$$g(\beta\rho) = f_1\beta\rho + \mathrm{O}\left(|\beta\rho|^2\right) \quad \text{for } |\beta\rho| \to 0, \ \mathrm{Re} \quad > -\delta, \tag{4.4}$$

with f_1 a finite constant, and so from (2.7d) we obtain

$$g(\beta\rho) = \mathrm{o}\left(|\beta\rho|^{\nu-1}|L(\beta\rho)|\right) \quad \text{for } |\rho| \to 0, \ \mathrm{Re}\,\rho \geqslant 0. \tag{4.5}$$

Consequently, the equation

$$cx^{\nu-1}L(x) = \frac{1-a}{a}, \quad x > 0, \ 0 < 1 - a \ll 1, \tag{4.6}$$

has a unique root $x = \Delta(a)$ such that

$$\Delta(a) \downarrow 0 \quad \text{for } a \uparrow 1; \tag{4.7}$$

note that the left-hand side of (4.6) is regular for $\mathrm{Re}\,x > 0$, continuous for $\mathrm{Re}\,x \geqslant 0$.

This root of (4.6) will be called the *coefficient of contraction*, and equation (4.6) will be called the *contraction equation.*

Obviously, we have

$$\lim_{a \uparrow 1} \frac{\Delta(a)}{\delta(a)} = 1. \tag{4.8}$$

From (4.6) we obtain

$$\frac{ac}{1-a}\Delta(a) = \left(\frac{1-a}{ac}\right)^{(2-\nu)/(\nu-1)} \left[L(\Delta(a))\right]^{-1/(\nu-1)}, \tag{4.9}$$

hence from (2.7d) and (4.7),

$$\lim_{a\uparrow 1} \frac{ac}{1-a}\Delta(a) = 0. \tag{4.10}$$

Further,

$$\frac{ac}{1-a}\left[\Delta(a)\right]^{\mu-1} = \left(\frac{1-a}{ac}\right)^{(\mu-\nu)/(\nu-1)} \left[L(\Delta(a))\right]^{-(\mu-1)/(\nu-1)}. \tag{4.11}$$

Hence from (2.7a), (2.7b), (2.7d), (2.8) and (4.11),

$$\lim_{a\uparrow 1} \frac{ac}{1-a}\left[\Delta(a)\right]^{\mu-1} = 0. \tag{4.12}$$

Remark 4.1. Note that (4.10) and (4.12) also hold if $\mu \geqslant \nu = 2$, cf. (2.7d) for $\nu = 2$.

Below we consider the equations for the coefficient of contraction for the five cases discussed in section 3.

Case (i)

We take for the sake of simplicity $\widehat{\delta} = 1$ so that

$$\beta = \frac{2-\nu}{\nu-1}\frac{1}{s}.$$

The equation for the coefficient of contraction reads

$$\frac{1}{2-\nu}\left(\frac{\nu-1}{2-\nu}\right)^{\nu-1} x^{\nu-1}\left[1 - \frac{\nu-1}{2-\nu}x\right]^{-2} = \frac{1-a}{a},$$
$$x > 0,\ 0 < 1-a \ll 1. \tag{4.13}$$

Obviously, we have

$$\Delta(a) \sim \Delta_1(a) := \frac{2-\nu}{\nu-1}\left[(2-\nu)\frac{1-a}{a}\right]^{1/(\nu-1)}, \quad \text{for } a \uparrow 1. \tag{4.14}$$

Case (ii)

Here the equation for the coefficient of contraction reads

$$\frac{c\pi}{\Gamma(\nu)\sin(\nu-1)\pi}x^{\nu-1} = \frac{1-a}{a}, \quad x > 0,\ 0 < 1-a \ll 1, \tag{4.15}$$

so that

$$\Delta(a) = \left[\frac{1-a}{ac}\frac{\Gamma(\nu)\sin(\nu-1)\pi}{\pi}\right]^{1/(\nu-1)}, \quad \text{for } 0 < 1-a \ll 1. \tag{4.16}$$

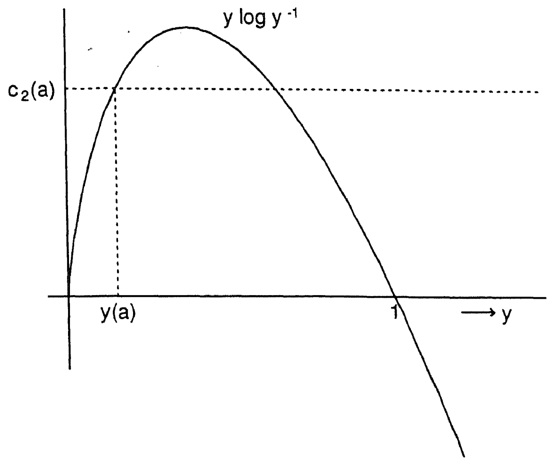

Figure 1.

For the case (3.8) it is readily seen that the equation for the coefficient of contraction reads

$$\frac{c\pi}{\Gamma(\nu)\sin(\nu-1)\pi}x^{\nu-1}\left\{1+\frac{c_1}{c}\frac{\Gamma(\nu)\sin(\nu-1)\pi}{\Gamma(\nu_1)\sin(\nu_1-1)\pi}x^{\nu_1-\nu}\right\}=\frac{1-a}{a}. \tag{4.17}$$

Obviously, the right-hand side of (4.16) is a first-order approximation for $a \uparrow 1$ of the zero $\Delta(a)$ of (4.17) which tends to zero for $a \uparrow 1$. Similarly for the case (3.9).

Case (iii)

For this case, cf. (3.11), the equation reads

$$\frac{c\pi}{\Gamma(\nu)\sin(\nu-1)\pi}x^{\nu-1}\log x^{-1}=\frac{1-a}{a}, \quad x>0,\ 0<1-a\ll 1. \tag{4.18}$$

Remark 4.2. Since we need the root $\Delta(a)$ of (4.18) which approaches zero from above for $a \uparrow 1$, equation (4.18) should be only considered for those values of $a \in (\frac{1}{2}, 1)$ for which

$$\frac{1-a}{ac}\frac{\Gamma(\nu)\sin(\nu-1)\pi}{\pi} < 1.$$

For the numerical solution of (4.18) put $y = x^{\nu-1}$, so that (4.18) transforms into

$$y\log y^{-1} = c_2(a),$$

with

$$c_2(a) \downarrow 0 \quad \text{for } a \uparrow 1,$$

and $y(a)$ $(= (\Delta(a))^{\nu-1})$ is that solution for which holds that $y(a) \downarrow 0$ for $a \uparrow 1$, see figure 1.

Case (iv)

The equation reads here

$$cx \log x^{-1} = \frac{1-a}{a}, \quad x > 0,\ 0 < 1 - a \ll 1. \tag{4.19}$$

It can be easily solved numerically.

Case (v)

The equation reads

$$cx \log x^{-1} + \frac{1}{2} cx^2 (\log \gamma x)^2 = \frac{1-a}{a}, \quad x > 0,\ 0 < 1 - a \ll 1. \tag{4.20}$$

It is readily seen that the zero $\Delta(a)$ of (4.20) with $\Delta(a) \downarrow 0$ for $a \uparrow 1$ has as a first-order approximation the zero $\Delta(a)$ of (4.19).

From the above examples of the equations for the coefficient of contraction it is seen that in general the determination of $\Delta(a)$ for a sufficiently close to one can only be done numerically; cf., e.g., equations (4.17), (4.20). However, a first-order approximation of $\Delta(a)$ is usually easy to obtain. In this respect the following result from the theory of regularly varying functions is very useful. In [3, pp. 334, 335], it is shown that the following are equivalent:

$$1 - B(t) \sim \frac{-1}{\Gamma(1-\nu)} \left(\frac{\beta}{t}\right)^{\nu} l\left(\frac{t}{\beta}\right), \quad t \to \infty, \quad 1 < \nu < 2, \tag{4.21}$$

with $l(t)$ a slowly varying function at infinity (cf. [3]), and

$$1 - \frac{1 - \beta\{\rho\}}{\beta\rho} \sim (\beta\rho)^{\nu-1} l\left(\frac{1}{\beta\rho}\right) \quad \text{for } |\rho| \downarrow 0,\ \rho > 0. \tag{4.22}$$

Hence, if in (2.3),

$$G_1(t) = \frac{-1}{\Gamma(1-\nu)} \left(\frac{\beta}{t}\right)^{\nu} l\left(\frac{t}{\beta}\right) \quad \text{for } t \to \infty, \quad 1 < \nu < 2, \tag{4.23}$$

then (4.22) holds. Consequently, it is seen that a first-order approximation of $\Delta(a)$ for $a \uparrow 1$ is given by that root of the equation

$$x^{\nu-1} l\left(\frac{1}{x}\right) = \frac{1-a}{a} \tag{4.24}$$

which tends to zero for $a \uparrow 1$. Interesting examples are here:

$$l\left(\frac{t}{\beta}\right) = \log\left(\frac{t}{\beta}\right)^{n}, \quad n \text{ a positive integer}, \tag{4.25a}$$

$$l\left(\frac{t}{\beta}\right) = \log \log t. \tag{4.25b}$$

5. The stationary waiting time distribution

The main goal of this section is to prove theorem 5.1, a heavy-traffic limit theorem for the waiting time distribution in the GI/G/1 queue with interarrival and service time distributions satisfying the (tail-) assumptions of section 2.

The GI/G/1 queueing model under consideration has a unique stationary waiting time distribution $W(t)$, say, since $a < 1$, cf. (2.1). Let $\mathbf{W}$ be a stochastic variable with distribution $W(t)$. Denote by $\mathbf{n}$ the number of customers served in a busy period, and by $\mathbf{i}$ the idle period. It is well known, cf. [10, p. 286], that

$$a < 1 \Leftrightarrow \mathrm{E}[\mathbf{n}] < \infty \Rightarrow \mathrm{E}[\mathbf{i}] = (\alpha - \beta)\mathrm{E}[\mathbf{n}], \tag{5.1}$$

and that, cf. [10, p. 371]: for $\operatorname{Re}\rho = 0$,

$$\mathrm{E}\big[\mathrm{e}^{-\rho\mathbf{W}}\big] = \frac{(\beta - \alpha)\rho}{1 - \beta\{\rho\}\alpha\{-\rho\}} \frac{1 - \mathrm{E}[\mathrm{e}^{\rho\mathbf{i}}]}{-\rho\mathrm{E}[\mathbf{i}]},$$

or

$$\omega\{\rho\} = \frac{(\beta - \alpha)\rho}{1 - \beta\{\rho\}\alpha\{-\rho\}}\chi\{-\rho\}, \quad \operatorname{Re}\rho = 0, \tag{5.2}$$

with: for $\operatorname{Re}\rho \geqslant 0$,

$$\omega\{\rho\} := \mathrm{E}\big[\mathrm{e}^{-\rho\mathbf{W}}\big], \qquad \chi\{\rho\} := \frac{1 - \mathrm{E}[\mathrm{e}^{-\rho\mathbf{i}}]}{\rho\mathrm{E}[\mathbf{i}]}. \tag{5.3}$$

$\omega\{\rho\}$ and $\chi\{\rho\}$ are regular for $\operatorname{Re}\rho > 0$, continuous for $\operatorname{Re}\rho \geqslant 0$, and $\omega\{0\} = 1$, $\chi\{0\} = 1$.

We first write: for $\operatorname{Re}\rho = 0$,

$$\begin{aligned}\frac{1 - \beta\{\rho\}\alpha\{-\rho\}}{(\beta - \alpha)\rho} &= 1 + \frac{a}{1-a}\left[1 - \frac{1 - \beta\{\rho\}}{\beta\rho}\right] - \frac{1}{1-a}\left[1 - \frac{1 - \alpha\{-\rho\}}{-\alpha\rho}\right] \\ &\quad + \frac{a}{1-a}\frac{[1 - \alpha\{-\rho\}][1 - \beta\{\rho\}]}{\beta\rho} \\ &= \left[1 + \frac{a}{1-a}\left[1 - \frac{1 - \beta\{\rho\}}{\beta\rho}\right]\right]\big[1 + F(\beta\rho, a)\big],\end{aligned} \tag{5.4}$$

with: for $\operatorname{Re}\rho = 0$,

$$F(\beta\rho, a) := \frac{-\frac{1}{1-a}\left[1 - \frac{1-\alpha\{-\rho\}}{-\alpha\rho}\right] + \frac{a}{1-a}[1 - \alpha\{-\rho\}][1 - \beta\{\rho\}]/\beta\rho}{1 + \frac{a}{1-a}\left[1 - \frac{1-\beta\{\rho\}}{\beta\rho}\right]}. \tag{5.5}$$

By starting from the relations (4.10) and (4.12), the following lemma has been proven in appendix B. We remind the reader that the function $g(\cdot)$, that is mentioned below, has been introduced in (2.6). For the definition of the coefficient of contraction $\Delta(a)$, see (4.6) and (4.7).

Lemma 5.1. For $\mathrm{Re}\, r = 0$, $r \neq 0$,

$$\text{(i)}\ \lim_{a\uparrow 1}\left|\frac{a}{1-a}g\big(r\Delta(a)\big)\right| = 0,$$

$$\text{(ii)}\ \lim_{a\uparrow 1}\left|\frac{a}{1-a}\big[1-\alpha\{-r\Delta(a)\}\big]\frac{1-\beta\{r\Delta(a)\}}{\beta r\Delta(a)}\right| = 0,$$

$$\text{(iii)}\ \lim_{a\uparrow 1}\left|\frac{1}{1-a}\left[1-\frac{1-\alpha\{-r\Delta(a)\}}{-\alpha r\Delta(a)}\right]\right| = 0,$$

$$\text{(iv)}\ \lim_{a\uparrow 1}\left(1+\frac{a}{1-a}\left[1-\frac{1-\beta\{r\Delta(a)\}}{\beta r\Delta(a)}\right]\right) = 1+(\beta r)^{\nu-1}.$$

From (5.5) and lemma 5.1 it is seen that, for $\mathrm{Re}\, r = 0$,

$$\lim_{a\uparrow 1} F\big(\beta r\Delta(a), a\big) = 0, \tag{5.6}$$

$$\lim_{a\uparrow 1}\frac{1-\beta\{r\Delta(a)\}\alpha\{-r\Delta(a)\}}{(\beta-\alpha)r\Delta(a)} = 1+(\beta r)^{\nu-1}. \tag{5.7}$$

Notice that the determination of $\omega\{\rho\}$ and $\chi\{\rho\}$ from (5.2) and the required properties formulated below (5.3) amounts to solving a Wiener–Hopf boundary value problem. We shall solve this boundary value problem for $0 < 1-a \ll 1$, and then consider the limit for $a \uparrow 1$ (see appendix C). First define, cf. (C.4), for $|\rho| < \infty$,

$$H(\rho) := \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left[\log\frac{1-\beta\{\xi\}\alpha\{-\xi\}}{(\beta-\alpha)\xi}\right]\frac{\rho\,\mathrm{d}\xi}{(\xi-\rho)\xi}. \tag{5.8}$$

Replacing ρ by $r\Delta(a)$ and ξ by $\eta\Delta(a)$ it follows from (5.7) (cf. appendix C) that

$$\lim_{a\uparrow 1} H\big(r\Delta(a)\big) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\big[\log\big(1+(\beta\eta)^{\nu-1}\big)\big]\frac{r\,\mathrm{d}\eta}{(\eta-r)\eta}. \tag{5.9}$$

Note that the last integral is a principal value singular Cauchy integral for $\mathrm{Re}\, r \neq 0$ at $\eta = 0$ and also a principal value singular Cauchy integral for $\mathrm{Re}\, r = 0$. The integral converges absolutely and the logarithm satisfies on intervals with finite endpoints the Hölder condition, because it is differentiable, except at $\eta = 0$, cf. [30, p. 13], and, further, the condition (26)iv of [11], since $0 < \nu - 1 < 1$.

It follows from [11, theorem 4] that the solution of the boundary value problem reads, for $0 < 1-a \ll 1$ and after taking $\rho = r\Delta(a)$ in (5.2),

$$\begin{aligned}\omega\big\{r\Delta(a)\big\} &= \mathrm{e}^{H(r\Delta(a))}, && \mathrm{Re}\, r > 0,\\ \chi\big\{-r\Delta(a)\big\} &= \mathrm{e}^{H(-r\Delta(a))}, && \mathrm{Re}\, r < 0.\end{aligned} \tag{5.10}$$

Now apply contour integration of the integral in (5.9) in the right-half, respectively left-half plane, cf. appendix C. Because $1+(\beta\eta)^{\nu-1}$ is regular for $\operatorname{Re}\eta > 0$, continuous for $\operatorname{Re}\eta \geqslant 0$, and

$$\left|1+(\beta\eta)^{\nu-1}\right| \sim (\beta R)^{\nu-1} \quad \text{for } \eta = R\mathrm{e}^{i\phi},\ |\phi| \leqslant \frac{1}{2}\pi,\ R \gg 1,$$

and $1 < \nu \leqslant 2$, the following limits exist and for $\operatorname{Re} r > 0$:

$$\widehat{\omega}\{r\} := \lim_{a\uparrow 1} \omega\left\{r\Delta(a)/\beta\right\} = \frac{1}{1+r^{\nu-1}}, \tag{5.11a}$$

$$\widehat{\chi}\{-r\} := \lim_{a\uparrow 1} \chi\left\{-r\Delta(a)/\beta\right\} = 1. \tag{5.11b}$$

From the continuity theorem for LST of probability distributions with support contained in $[0,\infty)$ it follows that $\widehat{\omega}\{r\}$ is the LST of a nondefective probability distribution $R_{\nu-1}(t)$:

$$\int_{0-}^{\infty} \mathrm{e}^{-rt}\,\mathrm{d}R_{\nu-1}(t) = \frac{1}{1+r^{\nu-1}}, \quad \operatorname{Re} r \geqslant 0; \tag{5.12}$$

note that the right-hand side in (5.11a) indeed tends to one for $|r| \to 0$, $\operatorname{Re} r \geqslant 0$. The distribution $R_{\nu-1}(t)$ is called the *Kovalenko distribution* in [22]. In [3] it is called the Mittag–Leffler law, for further details see [3, p. 391].

Consequently, $\Delta(a)\mathbf{W}/\beta$ converges in distribution for $a \uparrow 1$, with limiting distribution $R_{\nu-1}(t)$. For this distribution we have

$$\int_{0}^{\infty} \mathrm{e}^{-rt}\big(1 - R_{\nu-1}(t)\big)\,\mathrm{d}t = \frac{r^{\nu-2}}{1+r^{\nu-1}}, \quad \operatorname{Re} r \geqslant 0,\ r \neq 0. \tag{5.13}$$

By applying theorem 2 of [18, Vol. II, p. 175], it is readily seen that: for $t \geqslant 0$,

$$1 - R_{\nu-1}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{n(\nu-1)}}{\Gamma(n(\nu-1)+1)}. \tag{5.14}$$

It should be observed that $R_{\nu-1}(t) = E_{\nu-1}(-t^{\nu-1})$, with $E_{\nu-1}(t)$ the Mittag–Leffler function, cf. [19, Vol. 3, p. 206]. By applying theorem 2 of [18, Vol. II, p. 159], we obtain the following asymptotic series for $R_{\nu-1}(t)$, $1 < \nu < 2$. For $t \to \infty$ and every finite $H \in \{1,2,\ldots\}$,

$$\begin{aligned} 1 - R_{\nu-1}(t) &= \sum_{n=1}^{H} (-1)^{n-1} \frac{t^{-n(\nu-1)}}{\Gamma(1-n(\nu-1))} + \mathrm{O}\big(t^{-(H+1)(\nu-1)}\big) \\ &= \frac{1}{\pi} \sum_{n=1}^{H} (-1)^{n-1} \frac{\Gamma(n(\nu-1)) \sin n(\nu-1)\pi}{t^{n(\nu-1)}} + \mathrm{O}\big(t^{-(H+1)(\nu-1)}\big). \end{aligned} \tag{5.15}$$

In the last equality we have used the identity $1/\Gamma(1-z) = [\Gamma(z)\sin\pi z]/\pi$, see also below (3.11). Note that

$$R_2(t) = 1 - \mathrm{e}^{-t}, \quad t \geqslant 0.$$

From the analysis in this section it is seen that the following heavy-traffic limit theorem has been proved.

Theorem 5.1. For the stable GI/G/1 queue with interarrival and service time distributions $A(t)$ and $B(t)$ satisfying the conditions (2.1) and (2.3)–(2.8), the "contracted" waiting time $\Delta(a)\mathbf{W}/\beta$ converges in distribution for $a \uparrow 1$, the limiting distribution $R_{\nu-1}(t)$ is given by (5.14) and the coefficient of contraction $\Delta(a)$ is that root of the equation (4.6) with the property that $\Delta(a) \downarrow 0$ for $a \uparrow 1$.

Corollary 5.1. Theorem 5.1 holds for $\mu \geqslant \nu = 2$ and also for $1 < \nu < 2$, $\mu \geqslant \nu$, if $|L(\beta\rho)| \to \infty$ for $|\rho| \to 0$, $\mathrm{Re}\,\rho \geqslant 0$.

Proof. Because (4.10) and (4.12) also hold for $\mu \geqslant \nu = 2$, cf. remark 4.1, it is readily verified that lemma 5.1 and (5.10) both apply for $\mu \geqslant \nu = 2$, so (5.11) follows as before. Further, for $1 < \nu < 2$ the relation (4.10) always holds, but (4.12) with $\mu = \nu$ holds only if $|L(\beta\rho)| \to \infty$ for $|\rho| \to 0$, $\mathrm{Re}\,\rho \geqslant 0$. □

Remark 5.2. Clearly, a special case of theorem 5.1 occurs when one considers the M/G/1 queue in which $B(\cdot)$ satisfies the conditions of section 2. According to the Pollaczek–Khintchine formula (cf. [10, p. 255]) we have, for $\mathrm{Re}\,\rho \geqslant 0$,

$$\mathrm{E}\big[\mathrm{e}^{-\rho\mathbf{W}}\big] = \frac{1-a}{1 - a\frac{1-\beta\{\rho\}}{\beta\rho}}. \tag{5.16}$$

Hence, after some rewriting, for $\mathrm{Re}\,r \geqslant 0$,

$$\mathrm{E}\big[\mathrm{e}^{-r\Delta(a)\mathbf{W}/\beta}\big] = \frac{1}{1 + \frac{a}{1-a}\big(1 - \frac{1-\beta\{r\Delta(a)/\beta\}}{r\Delta(a)}\big)}. \tag{5.17}$$

The result of theorem 5.1 for the M/G/1 queue now follows from lemma 5.1 (iv).

It is also simply obtained from a limit theorem formulated in [22, p. 38], concerning geometrical sums of i.i.d. stochastic variables. Notice that the geometrical sum

$$(1-a)\sum_{n=0}^{\infty} a^n(\mathbf{X}_1 + \cdots + \mathbf{X}_n),$$

with $\mathbf{X}_1, \mathbf{X}_2, \ldots$ independent stochastic variables with common distribution the distribution of a residual service time (hence with LST $(1-\beta\{\rho\})/\beta\rho$), has the same distribution as $\mathbf{W}$. For detailed results concerning the M/G/1 queue with $B(\cdot)$ a Pareto-type tail see [12].

Remark 5.3. $R_{\nu-1}(t)$ has also turned up as limiting distribution in functional limit theorems for risk processes with heavy tails. Consider the classical model of risk theory, with claims occurring according to a compound Poisson process. It is known that the ruin probability, starting from a level x, equals the steady-state probability $\mathbf{P}[\mathbf{W} > x]$ in the M/G/1 queue; cf. [33, p. 86]. Furrer et al. [24] consider that classical risk model. They assume that the claim size distribution is heavy-tailed, in such a way that the sum of n claim sizes, after subtraction of the mean and appropriate scaling by a factor $n^{1/\alpha}$ times a slowly varying function, weakly converges to α-stable Lévy motion. They assume that $1 < \alpha < 2$ (viz., the claim sizes have infinite variance). Under additional conditions, they show that a sequence of risk processes, parametrized by n, weakly converges for $n \to \infty$, in the Skorokhod topology, to an α-stable Lévy motion with drift. They also discuss the weak convergence of functionals of the risk process, like the ruin probability. Notice that the above scaling bears a relation to the coefficient of contraction that we apply to obtain a heavy-traffic limit result for the single-server queue.

Furrer [23] extends the classical model of risk theory by adding an α-stable Lévy motion to the compound Poisson process. He derives an elegant expression for the probability of ruin starting from a level x. The α-stable Lévy motion turns out to give rise to the Mittag–Leffler function, which we have also encountered in (5.14). Furrer [23] subsequently discusses the case in which the claim sizes of the compound Poisson process are heavy-tailed; he considers the separate effects of these heavy tails and of the α-stable Lévy motion on the tail of the ruin probability function.

Remark 5.4. Building upon the present study, we have recently considered the workload process $\{v_t,\ t \geqslant 0\}$ of the GI/G/1 queue with heavy-tailed interarrival and/or service time distribution in heavy traffic [7,15,16]. In order to get a proper limiting process, we not only apply the same coefficient of contraction $\Delta(a)$, but we also *scale time* by a factor $\Delta_1(a) := (1-a)\Delta(a)$. It can be shown that $\Delta(a)v_{\tau/\Delta_1(a)}$ converges in distribution for $a \uparrow 1$, for every $\tau > 0$. It can further be shown that the thus scaled and contracted workload process converges weakly to the workload process of a queueing model of which the input is described by a stable Lévy motion if $1 < \nu < 2$ and by Brownian motion if $\nu = 2$ (with ν the index of the heaviest tail).

Remark 5.5. Theorem 5.1 opens possibilities for approximating the waiting time distribution in the GI/G/1 queue with heavy-tailed service time (and possibly also interarrival time) distribution. These possibilities are explored in [6]. The preliminary results in that study are most promising. For example, consider an M/G/1 queue with LST of the service time distribution given by (3.3) (hence $\nu = 3/2$). This distribution is sufficiently nice to allow one to determine $\mathbf{P}[\mathbf{W} > t]$ exactly. Following theorem 5.1, we approximate $\mathbf{P}[\mathbf{W} > t]$ by $1 - R_{1/2}(\Delta(a)t/\beta)$; note that

$$R_{1/2}(t) = 1 - \mathrm{e}^t \operatorname{Erfc}\big(\sqrt{t}\,\big), \quad t > 0, \tag{5.18}$$

with the complementary error function being defined by

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \mathrm{e}^{-u^2}\,\mathrm{d}u. \tag{5.19}$$

Take $\Delta(a) = \Delta_1(a) = ((1-a)/(2a))^2$, cf. (4.14). The above approximation is remarkably accurate, even if the traffic load a is not close to one. Comparison with $1 - W_{\mathrm{ex}}(t)$, the exact waiting time tail, shows that for $a = 0.9$ the approximation is off by less than one percent for $1 - W_{\mathrm{ex}}(t)$ less than 0.2, and by less than 0.3% for $1 - W_{\mathrm{ex}}(t)$ less than 0.1. For $a = 0.5$ those same percentages increase to 12% and 4%. Even for $a = 0.1$, a very light traffic situation, the approximation provides errors less than 10% from $t = 20$ on.

6. The GI/G/1 queue with heavy-tailed interarrival time distribution

In this section we consider the case that the interarrival time distribution has a heavy tail; a tail that is, moreover, heavier than the tail of the service time distribution. Concerning $A(\cdot)$ and $B(\cdot)$ we make similar assumptions as in (2.3)–(2.8), but now with $A(\cdot)$ and $B(\cdot)$ reversed. It is assumed that

$$1 - A(t) = J_1(t) + J_2(t), \quad t \geqslant \alpha. \tag{6.1}$$

Here $J_2(t)$ will be a function for which holds that: for a $\delta > 0$,

$$\int_0^{\infty} \mathrm{e}^{-\rho t} J_2(t)\,\mathrm{d}t \quad \text{exists for Re } \rho > -\delta, \tag{6.2}$$

and $\alpha\{\rho\}$ can be represented as: for $\mathrm{Re}\,\rho \geqslant 0$,

$$1 - \frac{1 - \alpha\{\rho\}}{\alpha\rho} = h(\alpha\rho) + c(\alpha\rho)^{\mu-1} L(\alpha\rho), \tag{6.3}$$

where:

$$c > 0 \text{ is a constant}; \tag{6.4a}$$

$$1 < \mu \leqslant 2; \tag{6.4b}$$

$$h(\alpha\rho) \text{ is a regular function of } \rho \text{ for } \mathrm{Re}\,\rho > -\delta,\ h(0) = 0; \tag{6.4c}$$

$$\begin{cases} L(\alpha\rho) \text{ is regular for } \mathrm{Re}\,\rho > 0, \text{ and continuous for } \mathrm{Re}\,\rho \geqslant 0, \\ \quad \text{except possibly at } \rho = 0, \\ L(\alpha\rho) \to b > 0 \quad \text{for } |\rho| \to 0,\ \mathrm{Re}\,\rho \geqslant 0, \text{ with } b = \infty \text{ if } \mu = 2, \\ \lim\limits_{x \downarrow 0} \dfrac{L(\alpha\rho x)}{L(x)} = 1 \quad \text{for } \mathrm{Re}\,\rho \geqslant 0,\ \rho \neq 0; \end{cases} \tag{6.4d}$$

$$\text{for a } \lambda \in (1, \mu){:} \quad \int_0^{\infty} t^{\lambda}\,\mathrm{d}A(t) < \infty. \tag{6.4e}$$

Concerning $B(t)$ it will be assumed that

$$N_\nu := \int_0^\infty t^\nu \, \mathrm{d}B(t) < \infty \quad \text{for a } \nu > \mu. \tag{6.5}$$

Note that c, δ, $L(\cdot)$ and b are only for convenience denoted by the same symbols as in (2.7).

The analysis for the present case is quite similar to that for the case described in section 2. In the next section the main points of the analysis for the present case will be discussed in so far as they differ from the heavy-tailed $B(\cdot)$ case.

7. A heavy-traffic limit theorem for the case of a heavy-tailed interarrival time distribution

We start from (5.4): for $\mathrm{Re}\,\rho = 0$,

$$\frac{1-\beta\{\rho\}\alpha\{-\rho\}}{(\beta-\alpha)\rho} = 1 - \frac{1}{1-a}\left[1 - \frac{1-\alpha\{-\rho\}}{-\alpha\rho}\right] + \frac{a}{1-a}\left[1 - \frac{1-\beta\{\rho\}}{\beta\rho}\right] + \frac{a}{1-a}\frac{[1-\alpha\{-\rho\}][1-\beta\{\rho\}]}{\beta\rho}. \tag{7.1}$$

We have: for $\mathrm{Re}\,\rho \leqslant 0$,

$$1 - \frac{1}{1-a}\left[1 - \frac{1-\alpha\{-\rho\}}{-\alpha\rho}\right] = \frac{1}{1-a}\left[-a + \frac{1-\alpha\{-\rho\}}{-\alpha\rho}\right]. \tag{7.2}$$

From GI/M/1 theory it is well-known that the equation

$$\frac{1-\alpha\{\rho\}}{\alpha\rho} = a, \quad \mathrm{Re}\,\rho \geqslant 0,\ a < 1, \tag{7.3}$$

has exactly one root $\rho = \lambda(a)/\alpha$; this root is real, positive, and $\lambda(a) \downarrow 0$ for $a \uparrow 1$. Hence it follows from (6.3) that $\lambda(a)$ is the only root of

$$h(x) + cx^{\mu-1}L(x) = 1-a, \quad \mathrm{Re}\,x \geqslant 0. \tag{7.4}$$

Denote, for $0 < 1-a \ll 1$, by $\Lambda(a)$ that unique root of

$$cx^{\mu-1}L(x) = 1-a, \quad x \geqslant 0, \tag{7.5}$$

which tends to zero for $a \uparrow 1$. Obviously, we have

$$\lim_{a\uparrow 1} \frac{\Lambda(a)}{\lambda(a)} = 1. \tag{7.6}$$

For the present case $\Lambda(a)$ will be called the *coefficient of contraction*, and equation (7.5) the *contraction equation*.

Put $\rho = r\Lambda(a)$, then from (7.2) and (6.3) we have: for $\operatorname{Re} r = 0$,

$$1 - \frac{1}{1-a}\left[1 - \frac{1-\alpha\{-r\Lambda(a)\}}{-\alpha r\Lambda(a)}\right]$$
$$= \frac{1}{1-a}\left[1 - a - c(-\alpha r)^{\mu-1}\big(\Lambda(a)\big)^{\mu-1}\frac{L(-r\alpha\Lambda(a))}{L(\Lambda(a))}L\big(\Lambda(a)\big) - h\big(-\alpha r\Lambda(a)\big)\right]. \tag{7.7}$$

So from (7.5) and the conditions on $h(\cdot)$ and $L(\cdot)$ in section 6 we obtain: for $\operatorname{Re} r = 0$,

$$\lim_{a\uparrow 1} 1 - \frac{1}{1-a}\left[1 - \frac{1-\alpha\{-r\Lambda(a)\}}{-\alpha r\Lambda(a)}\right] = 1 - (-\alpha r)^{\mu-1}. \tag{7.8}$$

A similar analysis as in section 5 shows that for $a \uparrow 1$ the last two terms in (7.1) with $\alpha\rho = \alpha r\Lambda(a)$, $\operatorname{Re} r = 0$, both tend to zero for $a \uparrow 1$.

Next we consider the boundary value problem for the present case. It follows from (5.2) that

(i) $\omega\{r\Lambda(a)\}(1+\alpha r) = \dfrac{(\beta-\alpha)r\Lambda(a)(1+\alpha r)}{1-\beta\{r\Lambda(a)\}\alpha\{-r\Lambda(a)\}}\chi\{-r\Lambda(a)\}, \quad \operatorname{Re} r = 0,$ (7.9)

(ii) $\omega\{r\Lambda(a)\}$ and $\chi\{r\Lambda(a)\}$ are both regular for $\operatorname{Re} r > 0$, continuous for $\operatorname{Re} r \geqslant 0$,

(iii) $|\omega\{r\Lambda(a)\}| \leqslant 1$, $|\chi\{r\Lambda(a)\}| \leqslant 1$, for $\operatorname{Re} r \geqslant 0$, and $\omega\{0\} = \chi\{0\} = 1$.

In appendix D it is shown that this boundary value problem has a unique solution apart from a constant factor D_1 if the first factor in the right-hand side of (7.9) satisfies certain conditions, cf. (D.2) of appendix D. As in appendix C it is shown by using (7.8) that for $0 < 1 - a \ll 1$ these conditions are satisfied, and that the following limits exist:

$$(1+\alpha r)\widehat{\omega}\{r\} := \lim_{a\uparrow 1}(1+\alpha r)\omega\{r\Lambda(a)\} = D_1, \quad \operatorname{Re} r > 0, \tag{7.10}$$

$$\widehat{\chi}\{-r\} := \lim_{a\uparrow 1}\chi\{-r\Lambda(a)\}$$
$$= D_1 \exp\left[\frac{-1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left[\log\frac{1+\alpha\eta}{1-(-\alpha\eta)^{\mu-1}}\right]\frac{r\,\mathrm{d}\eta}{(\eta-r)\eta}\right],$$
$$\operatorname{Re} r < 0. \tag{7.11}$$

Because $\widehat{\omega}\{0\} = 1$ we obtain

$$D_1 = 1. \tag{7.12}$$

Hence we find: for $\operatorname{Re} r \geqslant 0$,

$$\widehat{\omega}\{r/\alpha\} = \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\Lambda(a)\mathbf{W}/\alpha}\} = \frac{1}{1+r}. \tag{7.13}$$

We finally determine $\hat{\chi}\{r/\alpha\}$: for $\mathrm{Re}\, r \geqslant 0$,

$$\hat{\chi}\{r/\alpha\} = 1. \tag{7.14}$$

The latter result follows by considering the following principal value singular Cauchy integral: for $\mathrm{Re}\, r < 0$,

$$-\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \left[\log \frac{1+\alpha\eta}{1-(-\alpha\eta)^{\mu-1}}\right] \frac{r\,\mathrm{d}\eta}{(\eta - r)\eta}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \left[\log \frac{1-(\alpha\zeta)^{\mu-1}}{1-\alpha\zeta}\right] \frac{r\,\mathrm{d}\zeta}{(\zeta + r)\zeta} = 0.$$

The latter integral has been closed by taking a large semicircle in the right-half plane, after which Cauchy's theorem has been applied.

The analysis given above leads to the following:

Theorem 7.1. For the stable GI/G/1 queue with interarrival and service time distributions $A(t)$ and $B(t)$ satisfying the conditions (2.1), (6.1)–(6.5), the "contracted" waiting time $\Lambda(a)\mathbf{W}/\alpha$ converges in distribution for $a \uparrow 1$, the limiting distribution is the negative exponential distribution with mean one, and the coefficient of contraction $\Lambda(a)$ is that root of the equation (7.5) with the property that $\Lambda(a) \downarrow 0$ for $a \uparrow 1$.

Remark 7.1. It is well known (cf. [10, p. 230]) that the waiting time distribution in the GI/M/1 queue is given by

$$\mathbf{P}[\mathbf{W} > t] = \lambda_0 \mathrm{e}^{-(1-\lambda_0)t/\beta}, \quad t > 0, \tag{7.15}$$

with λ_0 the smallest zero, in absolute value, of $z - \alpha\{(1-z)/\beta\}$. Obviously, cf. (7.3), $\lambda(a)/\alpha = (1-\lambda_0)/\beta$. This immediately yields theorem 7.1 in the GI/M/1 case.

Acknowledgements

The authors are indebted to S. Asmussen and Z. Michna for drawing their attention to [23,24], and to Mr. R. van der Horst for providing the figure and performing numerical calculations.

Appendix A

In this appendix we calculate the integral

$$I(\beta\rho) := \int_{\beta}^{\infty} \mathrm{e}^{-\rho t} \left(\frac{\beta}{t}\right)^2 \log \frac{t}{\beta} \frac{\mathrm{d}t}{\beta} = \int_{1}^{\infty} \mathrm{e}^{-\beta\rho t} \frac{\log t}{t^2}\, \mathrm{d}t, \quad \mathrm{Re}\, \rho \geqslant 0. \tag{A.1}$$

We have

$$I(\beta\rho)=\mathrm{e}^{-\beta\rho t}\left[-\frac{\log t}{t}-\frac{1}{t}\right]\Bigg|_1^\infty-\beta\rho\int_1^\infty \mathrm{e}^{-\beta\rho t}\left[\frac{\log t}{t}+\frac{1}{t}\right]\mathrm{d}t$$
$$=\mathrm{e}^{-\beta\rho}-\beta\rho\int_1^\infty \mathrm{e}^{-\beta\rho t}\frac{\mathrm{d}t}{t}-\beta\rho\int_1^\infty \mathrm{e}^{-\beta\rho t}\frac{\log t}{t}\,\mathrm{d}t. \tag{A.2}$$

From [19, Vol. 2, p. 144], we have: for $\mathrm{Re}\,\rho \geqslant 0$,

$$I_1(\beta\rho) := \beta\rho\int_1^\infty \mathrm{e}^{-\beta\rho t}\frac{\mathrm{d}t}{t} = -\gamma\beta\rho-\beta\rho\log\beta\rho+\sum_{n=1}^{\infty}\frac{(-\beta\rho)^{n+1}}{n!n}, \tag{A.3}$$

with γ Euler's constant. Further, partial integration yields

$$I_2(\beta\rho) := \beta\rho\int_1^\infty \mathrm{e}^{-\beta\rho t}\frac{\log t}{t}\,\mathrm{d}t = \frac{1}{2}\beta\rho\int_1^\infty \mathrm{e}^{-\beta\rho t}\frac{\mathrm{d}}{\mathrm{d}t}(\log t)^2\,\mathrm{d}t$$
$$=\frac{1}{2}(\beta\rho)^2\int_1^\infty \mathrm{e}^{-\beta\rho t}(\log t)^2\,\mathrm{d}t. \tag{A.4}$$

From [20, Vol. 1, p. 149], we have: for $\mathrm{Re}\,\rho > 0$,

$$I_2(\beta\rho)=\frac{1}{2}(\beta\rho)^2\int_0^\infty \mathrm{e}^{-\beta\rho t}(\log t)^2\,\mathrm{d}t-\frac{1}{2}(\beta\rho)^2\int_0^1 \mathrm{e}^{-\beta\rho t}(\log t)^2\,\mathrm{d}t$$
$$=\frac{1}{2}\beta\rho\left[\frac{1}{6}\pi^2+\beta\rho(\log\gamma\beta\rho)^2\right]-\frac{1}{2}(\beta\rho)^2\int_0^1 \mathrm{e}^{-\beta\rho t}(\log t)^2\,\mathrm{d}t. \tag{A.5}$$

Note that

$$\int_0^u (\log t)^2\,\mathrm{d}t = u(\log u)^2-2u\log u+2u,$$

which shows that the last integral exists for all ρ. Hence we obtain from the above relations: for $\mathrm{Re}\,\rho \geqslant 0$,

$$I(\beta\rho)=h_2(\beta\rho)+\beta\rho\log\beta\rho-\frac{1}{2}(\beta\rho)^2(\log\gamma\beta\rho)^2, \tag{A.6}$$

with

$$h_2(\beta\rho)=\mathrm{e}^{-\beta\rho}+\beta\rho\left(\gamma-\frac{1}{12}\pi^2\right)+\frac{1}{2}(\beta\rho)^2\int_0^1 \mathrm{e}^{-\beta\rho t}(\log t)^2\,\mathrm{d}t-\sum_{n=1}^{\infty}\frac{(-\beta\rho)^{n+1}}{n!n}. \tag{A.7}$$

Obviously, $h_2(\beta\rho)$ is an entire function of ρ.

Appendix B

In this appendix we prove lemma 5.1. Statement (i) follows from (4.4) and (4.10). For $\mathrm{Re}\,\rho \leqslant 0$,

$$\frac{1-\alpha\{-\rho\}}{-\alpha\rho} = \int_0^\infty \mathrm{e}^{\rho t}\big(1-A(t)\big)\frac{\mathrm{d}t}{\alpha},$$

so that, for $\mathrm{Re}\,r = 0$,

$$\left|\frac{1-\alpha\{-r\Delta(a)\}}{-\alpha r\Delta(a)}\right| \leqslant 1$$

(and similarly for the corresponding β-term). By using (4.10) we have that: for $\mathrm{Re}\,r = 0$, $r \neq 0$,

$$\left|\frac{a}{1-a}\alpha r\Delta(a)\frac{1-\alpha\{-r\Delta(a)\}}{-r\Delta(a)}\,\frac{1-\beta\{r\Delta(a)\}}{\beta r\Delta(a)}\right| \to 0 \quad \text{for } a \uparrow 1,$$

and (ii) has been proved.

From [29, p. 199], and (2.8) we have: for $\mu \leqslant 2$ and $\mathrm{Re}\,r = 0$, $r \neq 0$,

$$\alpha\{-r\} = 1 + \alpha r + f_2 M_\mu |r^\mu| \quad \text{for } |r| \to 0, \tag{B.1}$$

with f_2 a finite constant. Hence, by using (B.1) and (4.12): for $\mathrm{Re}\,r = 0$, $r \neq 0$,

$$\left|\frac{1}{1-a}\left[1-\frac{1-\alpha\{-r\Delta(a)\}}{-\alpha r\Delta(a)}\right]\right| = \frac{f_2 M_\mu}{\alpha}|r|^{\mu-1}\frac{1}{1-a}\Delta(a)^{\mu-1} \to 0,$$

which proves (iii) for $\mu \leqslant 2$; for $\mu > 2$ the statement follows again by using [29, p. 199].

Finally, from (2.6), (2.7d), (4.6) and lemma 5.1(i): for $\mathrm{Re}\,r = 0$, $r \neq 0$, and for $a \uparrow 1$,

$$\frac{a}{1-a}\left[1-\frac{1-\beta\{r\Delta(a)\}}{\beta r\Delta(a)}\right] = \frac{a}{1-a}g\big(\beta r\Delta(a)\big) + (\beta r)^{\nu-1}\frac{L(\beta r\Delta(a))}{L(\Delta(a))} \to (\beta r)^{\nu-1},$$

which proves (iv).

Appendix C

In this appendix we discuss the contour integration of (5.9). Let τ and σ be stochastic variables with distribution $B(\cdot)$ and $A(\cdot)$, respectively. Then with $\widehat{\gamma} := \min(\lambda, \mu)$ so that, cf. (2.7e) and (2.8),

$$1 < \widehat{\gamma} < 2, \tag{C.1}$$

we have, cf. [29, p. 155], that

$$\mathrm{E}\{|\tau-\sigma|^{\hat{\gamma}}\} < f_0\big[\mathrm{E}\{\tau^{\hat{\gamma}}\} + \mathrm{E}\{\sigma^{\hat{\gamma}}\}\big] < \infty, \tag{C.2}$$

with f_0 a finite constant.

With $\mathbf{n}$ the number of customers served in a busy period we have, cf. (2.1),

$$a < 1 \Rightarrow \mathrm{E}[\mathbf{n}] < \infty \quad \text{and} \quad \mathrm{E}[\mathbf{i}] = (\alpha - \beta)\mathrm{E}[\mathbf{n}]. \tag{C.3}$$

Put for $|\rho| < \infty$,

$$H(\rho) := \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \left[\log \frac{1 - \beta\{\xi\}\alpha\{-\xi\}}{(\beta - \alpha)\xi}\right] \frac{\rho\,\mathrm{d}\xi}{(\xi - \rho)\xi}. \tag{C.4}$$

In [11, appendix A, lemma 3], it is shown that the integral in (C.4) is well defined as a principal value Cauchy integral because of (C.2) and (C.3). Note that, cf. [11], with $D(R) := \{\xi\colon\ |\xi| \geqslant R,\ \mathrm{Re}\,\xi = 0\}$,

$$\left|\frac{1}{2\pi\mathrm{i}} \int_{D(R)} \left[\log \frac{1 - \beta\{\xi\}\alpha\{-\xi\}}{(\beta - \alpha)\xi}\right] \frac{\rho\,\mathrm{d}\xi}{(\xi - \rho)\xi}\right| \to 0 \quad \text{for } R \to \infty. \tag{C.5}$$

Because (C.2) holds, it is seen that the logarithm in (C.4) satisfies the Hölder condition with index $\widehat{\gamma} - 1$ on every interval of the imaginary axis with finite endpoints. The condition (26)iv of [11] plays the role of the Hölder conditions on intervals $(\mathrm{i}h_1, \mathrm{i}h_2)$ of the imaginary axis with $|h_1|$ and $|h_2|$ both large. For the present case, this condition reads

$$\left|\frac{1 - \beta\{\xi_1\}\alpha\{-\xi_1\}}{(\beta - \alpha)\xi_1} - \frac{1 - \beta\{\xi_2\}\alpha\{-\xi_2\}}{(\beta - \alpha)\xi_2}\right| \leqslant f_1 \left|\frac{1}{|\xi_1|^{\delta_1}} - \frac{1}{|\xi_2|^{\delta_1}}\right|, \tag{C.6}$$

for $|\xi_1|$ and $|\xi_2|$ both large, for a $\delta_1 \in (0, 1]$, and f_1 a constant. Putting in (C.6)

$$\xi_1 = \eta_1 \Delta(a), \quad \xi_2 = \eta_2 \Delta(a), \quad \mathrm{Re}\,\eta_1 = \mathrm{Re}\,\eta_2 = 0,$$

it is seen that for $0 < 1 - a \ll 1$, i.e., for $0 < \Delta(a) \ll 1$, the relation (C.6) applies because of (5.7).

With

$$\rho = r\Delta(a) \quad \text{and} \quad \xi = \eta\Delta(a),$$

we have: for $R > 0$, $|r| < \infty$, $\mathrm{Re}\,\eta = 0$,

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}R}^{\mathrm{i}R} \left[\log \frac{1 - \beta\{\xi\}\alpha\{-\xi\}}{(\beta - \alpha)\xi}\right] \frac{\rho\,\mathrm{d}\xi}{(\xi - \rho)\xi} \\ &\quad = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}R/\Delta(a)}^{\mathrm{i}R/\Delta(a)} \left[\log \frac{1 - \beta\{\eta\Delta(a)\}\alpha\{-\eta\Delta(a)\}}{(\beta - \alpha)\eta\Delta(a)}\right] \frac{r\,\mathrm{d}\eta}{(\eta - r)\eta}. \end{aligned} \tag{C.7}$$

Hence from (5.7) and the absolute and uniform convergence of the integral in (C.4): for $|r| < \infty$,

$$\lim_{a \uparrow 1} H\big(r\Delta(a)\big) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \left[\log\big(1 + (\beta\eta)^{\nu-1}\big)\right] \frac{r\,\mathrm{d}\eta}{(\eta - r)\eta}. \tag{C.8}$$

We shall now perform the contour integration in (C.8) (= (5.9)). Note that the integral in (C.8) is a principal value singular Cauchy integral, cf. [30, pp. 27, 28], or [17, section I.1.5]. It is defined by

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\log\left[1+(\beta\eta)^{\nu-1}\right]\frac{r\,\mathrm{d}\eta}{(\eta-r)\eta}$$
$$:=\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi \mathrm{i}}\int_{D(\varepsilon,r)}\log\left[1+(\beta\eta)^{\nu-1}\right]\frac{r\,\mathrm{d}\eta}{(\eta-r)\eta},$$

with $D(\varepsilon, R)$ the line segment:

$$D(\varepsilon,R):=\{\eta\colon\ \varepsilon\leqslant|\eta|\leqslant R,\ \mathrm{Re}\,\eta=0\}.$$

Denote by $C(\varepsilon, R)$ the contour

$$C(\varepsilon,R):=D(\varepsilon,R)\cup\left\{\eta\colon\ \eta=\varepsilon \mathrm{e}^{\mathrm{i}\phi},\ |\phi|\leqslant\frac{1}{2}\pi\right\}\cup\left\{\eta\colon\ \eta=R\mathrm{e}^{\mathrm{i}\phi},\ |\phi|\leqslant\frac{1}{2}\pi\right\}.$$

The calculation of the integral in (C.8) proceeds as follows. Application of Cauchy's theorem to $C(\varepsilon, R)$ yields (notice that integration is clockwise, and that the first term in the right-hand side below stems from the residue at the pole in r, for $\mathrm{Re}\,r > 0$)

$$\frac{1}{2\pi \mathrm{i}}\int_{D(\varepsilon,R)}\log\left[1+(\beta\eta)^{\nu-1}\right]\frac{r\,\mathrm{d}\eta}{(\eta-r)\eta}$$
$$=-\log\left[1+(\beta r)^{\nu-1}\right]-\frac{1}{2\pi \mathrm{i}}\int_{-\pi/2}^{\pi/2}\log\left[1+(\beta\eta)^{\nu-1}\right]\frac{r\,\mathrm{d}\eta}{(\eta-r)\eta}\bigg|_{\eta=\varepsilon \mathrm{e}^{\mathrm{i}\phi}}$$
$$+\frac{1}{2\pi \mathrm{i}}\int_{-\pi/2}^{\pi/2}\log\left[1+(\beta\eta)^{\nu-1}\right]\frac{r\,\mathrm{d}\eta}{(\eta-r)\eta}\bigg|_{\eta=R\mathrm{e}^{\mathrm{i}\phi}},$$

with $\varepsilon < |r| < R$. Here the last integral converges to zero for $R \to \infty$ because $1 < \nu \leqslant 2$. The second integral converges for $\varepsilon \downarrow 0$ also to zero, because $\log[1 + (\beta\eta)^{\nu-1}]$ is zero for $\eta = 0$, cf. [30, pp. 27, 28], or [17, section I.1.5]. From the last two relations it is readily seen that $-\log[1+(\beta r)^{\nu-1}]$ is the value of the integral in (C.8) (=(5.9)) for $\mathrm{Re}\,r > 0$. The result in (5.11b), for $\mathrm{Re}\,r < 0$, follows also from contour integration. For $\mathrm{Re}\,r = 0$ the result follows by applying the Plemelj–Sokhotski formula, cf. [17].

Appendix D

Consider the boundary value problem

$$\Phi(\rho)=K(\rho)\Omega(\rho),\quad \mathrm{Re}\,\rho=0; \tag{D.1a}$$

$$\begin{cases}\Phi(\rho)\text{ is regular for }\mathrm{Re}\,\rho>0,\text{ continuous for }\mathrm{Re}\,\rho\geqslant 0,\\ \Omega(\rho)\text{ is regular for }\mathrm{Re}\,\rho<0,\text{ continuous for }\mathrm{Re}\,\rho\leqslant 0;\end{cases} \tag{D.1b}$$

$$\begin{cases}|\Phi(\rho)/\rho|<\infty\quad\text{for }|\rho|\to\infty,\quad \mathrm{Re}\,\rho\geqslant 0,\\ |\Omega(\rho)|<\infty\quad\text{for }|\rho|\to\infty,\quad \mathrm{Re}\,\rho\leqslant 0.\end{cases} \tag{D.1c}$$

Concerning $K(\rho)$ the following is assumed. The integral

$$\Psi(\rho) := \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log K(\xi) \frac{\rho\, \mathrm{d}\xi}{(\xi - \rho)\xi} \tag{D.2}$$

with $|\Psi(\rho)| < \infty$ for $|\rho| \to \infty$ exists as a principal value singular Cauchy integral and: for $\mathrm{Re}\,\rho = 0$,

$$\Psi^+(\rho) := \lim_{z \to \rho;\ \mathrm{Re}\, z < 0} \Psi(z) = \frac{1}{2} \log K(\rho) + \Psi(\rho), \tag{D.3}$$

$$\Psi^-(\rho) := - \lim_{z \to \rho;\ \mathrm{Re}\, z > 0} \Psi(z) = -\frac{1}{2} \log K(\rho) + \Psi(\rho). \tag{D.4}$$

It then follows that: for $\mathrm{Re}\,\rho = 0$,

$$\Phi(\rho)\mathrm{e}^{\Psi^-(\rho)} = \Omega(\rho)\mathrm{e}^{\Psi^+(\rho)}. \tag{D.5}$$

From (D.2) it is seen that $\Psi(\rho)$ is regular for $\mathrm{Re}\,\rho < 0$ as well as for $\mathrm{Re}\,\rho > 0$, and further that, cf. (D.1b) and (D.4), $\Phi(\rho)\exp[\Psi^-(\rho)]$, $\mathrm{Re}\,\rho = 0$, is the boundary value of a regular function in $\mathrm{Re}\,\rho > 0$, and $\Omega(\rho)\exp[\Psi^+(\rho)]$, $\mathrm{Re}\,\rho = 0$, is the boundary value of a regular function in $\mathrm{Re}\,\rho < 0$. Hence the functions $\Phi(\rho)\exp[\Psi(\rho)]$, $\mathrm{Re}\,\rho > 0$ and $\Omega(\rho)\exp[\Psi(\rho)]$, $\mathrm{Re}\,\rho < 0$ are each other's analytic continuations. Because $|\Psi(\rho)| < \infty$ for $|\rho| \to \infty$ it follows from (D.1c) and Liouville's theorem that

$$\begin{aligned} \Phi(\rho)\mathrm{e}^{\Psi(\rho)} &= D_1 + \rho D_2, \quad \mathrm{Re}\,\rho > 0, \\ \Omega(\rho)\mathrm{e}^{\Psi(\rho)} &= D_1 + \rho D_2, \quad \mathrm{Re}\,\rho < 0, \end{aligned} \tag{D.6}$$

or

$$\begin{aligned} \Phi(\rho) &= \begin{cases} (D_1 + \rho D_2)\mathrm{e}^{-\Psi(\rho)}, & \mathrm{Re}\,\rho > 0, \\ (D_1 + \rho D_2)\mathrm{e}^{-\Psi^-(\rho)}, & \mathrm{Re}\,\rho = 0, \end{cases} \\ \Omega(\rho) &= \begin{cases} (D_1 + \rho D_2)\mathrm{e}^{-\Psi(\rho)}, & \mathrm{Re}\,\rho < 0, \\ (D_1 + \rho D_2)\mathrm{e}^{-\Psi^+(\rho)}, & \mathrm{Re}\,\rho = 0. \end{cases} \end{aligned} \tag{D.7}$$

From (D.1c) and (D.2) it follows that we should have $D_2 = 0$. Actually, (D.7) with $D_2 = 0$ is the unique solution of the boundary value problem (D.1) under the conditions assumed to hold for $K(\rho)$.

References

[1] J. Abate, G.L. Choudhury and W. Whitt, Calculation of the GI/G/1 waiting time distribution and its cumulants from Pollaczek's formulas, Arch. Elektr. Uebertragung 47 (Pollaczek memorial volume, 1993) 311–321.

[2] J. Beran, R. Sherman, M.S. Taqqu and W. Willinger, Long-range dependence in variable-bit-rate video, IEEE Trans. Commun. 43 (1995) 1566–1579.

[3] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation* (Cambridge University Press, Cambridge, 1987).
[4] O.J. Boxma, Fluid queues and regular variation, Performance Evaluation 27/28 (1996) 699–712.
[5] O.J. Boxma, Regular variation in a multi-source fluid queue, in: *Proceedings of ITC-15*, eds. R. Ramaswami and P. Wirth (North-Holland, Amsterdam, 1997) pp. 391–402.
[6] O.J. Boxma and J.W. Cohen, The M/G/1 queue with heavy-tailed service time distribution, IEEE J. Selected Areas Commun. 16 (1998) 749–763.
[7] O.J. Boxma and J.W. Cohen, The single server queue: Heavy tails and heavy traffic, in: *Self-Similar Network Traffic and Performance Evaluation*, eds. K. Park and W. Willinger (Wiley, New York, 1999).
[8] J.W. Cohen, Some results on regular variation for distributions in queueing and fluctuation theory, J. Appl. Probab. 10 (1973) 343–353.
[9] J.W. Cohen, Superimposed renewal processes and storage with gradual input, Stochastic Process. Appl. 2 (1974) 31–58.
[10] J.W. Cohen, *The Single Server Queue* (North-Holland, Amsterdam, revised edition, 1982).
[11] J.W. Cohen, Complex functions in queueing theory, Archiv Elektr. Uebertragung 47 (Pollaczek memorial volume, 1993) 300–311.
[12] J.W. Cohen, On the M/G/1 queue with heavy-tailed service time distributions, Technical Report, PNA-R9702, CWI, Amsterdam (1997).
[13] J.W. Cohen, A heavy-traffic theorem for the GI/G/1 queue with a Pareto-type service time distribution, J. Appl. Math. Stochastic Anal. 11 (1997) 247–254.
[14] J.W. Cohen, Heavy-traffic limit theorems for the heavy-tailed GI/G/1 queue, Technical Report, PNA-R9719, CWI, Amsterdam (1997).
[15] J.W. Cohen, Heavy-traffic theory for the heavy-tailed M/G/1 queue and ν-stable Lévy noise traffic, Technical Report, PNA-R9805, CWI, Amsterdam (1998).
[16] J.W. Cohen, The ν-stable Lévy motion in heavy-traffic analysis of queueing models with heavy-tailed distributions, Technical Report, PNA-R9808, CWI, Amsterdam (1998).
[17] J.W. Cohen and O.J. Boxma, *Boundary Value Problems in Queueing System Analysis* (North-Holland, Amsterdam, 1983).
[18] G. Doetsch, *Handbuch der Laplace Transformation*, Vols. I–III (Birkhäuser, Basel, 1950).
[19] A. Erdélyi et al., *Higher Transcendental Functions*, Vols. 1–3 (McGraw-Hill, New York, 1953).
[20] A. Erdélyi et al., *Tables of Integral Transforms*, Vols. 1, 2 (McGraw-Hill, New York, 1954).
[21] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II (Wiley, New York, 1971).
[22] B.V. Gnedenko and V.Yu. Korolev, *Random Summation* (CRC Press, Boca Raton, FL, 1996).
[23] H. Furrer, Risk processes perturbed by α-stable Lévy motion, Report, Department of Mathematics, ETHZ (1996).
[24] H. Furrer, Z. Michna and A. Weron, Stable Lévy motion approximation in collective risk theory, Report, Department of Mathematical Statistics, Lund University; to appear in Insurance Mathematics and Economics (1996).
[25] O. Kella and W. Whitt, A storage model with a two-state random environment, Oper. Res. S40 (1992) S257–S262.
[26] P.R. Jelenković and A.A. Lazar, Multiplexing on/off sources with subexponential on-periods, CTR Technical Report 457-96-23, Columbia University, New York (1996); to appear in Adv. Appl. Probab. 31 (1999).
[27] J.F.C. Kingman, The heavy traffic approximation in the theory of queues, in: *Proceedings of the Symposium on Congestion Theory*, eds. W.L. Smith and W.E. Wilkinson (The University of North Carolina Press, Chapel Hill, 1965) pp. 137–159.
[28] W.E. Leland, M.S. Taqqu, W. Willinger and D.V. Wilson, On the self-similar nature of Ethernet traffic (extended version), IEEE/ACM Trans. Networking 2 (1994) 1–15.

[29] M. Loeve, *Probability Theory* (D. van Nostrand, New York, 2nd ed., 1960).
[30] N.I. Mushkelishvili, *Singular Integral Equations* (Noordhoff, Groningen, 1953).
[31] Z. Nehari, *Conformal Mapping* (Dover, New York, 1952).
[32] V. Paxson and S. Floyd, Wide area traffic: the failure of Poisson modeling, IEEE/ACM Trans. Networking 3 (1995) 226–244.
[33] N.U. Prabhu, *Stochastic Storage Processes* (Springer, New York, 1980).
[34] W. Whitt, Heavy traffic limit theorems for queues: a survey, in: *Mathematical Methods in Queueing Theory*, ed. A.B. Clarke (Springer, Berlin, 1974) pp. 307–350.
[35] W. Willinger, M.S. Taqqu, W.E. Leland and D.V. Wilson, Self-similarity in high-speed packet traffic: analysis and modeling of Ethernet traffic measurements, Statist. Sci. 10 (1995) 67–85.